Quantum Fingerprints of Classical Ruelle-Pollicott Resonances

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Quantum and classical correlations are studied experimentally in model n-disk microwave billiards. The wave vector κ autocorrelation $C(\kappa)$ of the quantum spectrum displays nonuniversal oscillations for large κ, comparable to the universal random matrix theory behavior observed for small κ. The nonuniversal behavior is shown to be completely determined by the classical Ruelle-Pollicott resonances, arising from the complex eigenvalues of the Perron-Frobenius operator, and calculated using periodic orbit theory. This work establishes a fundamental connection between the quantum and classical correlations of an open system.

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The quantum-classical correspondence for chaotic systems has been studied extensively in the context of universality and periodic orbit contributions. This approach has focused on eigenvalues and eigenfunctions and their statistical properties. Universality has been shown to arise from random matrix theory (RMT) [1], while periodic orbit contributions have been analyzed in the semiclassical scheme for calculations of eigenvalue spectra [2] and constructions of eigenfunctions [3].

An entirely different approach is to consider correlations of observables. In the classical context a probabilistic approach is best taken with Liouvillian dynamics. In certain classical systems these have been shown to lead to Ruelle-Pollicott (RP) resonances [4,5], arising from complex eigenvalues of the Perron-Frobenius operator. In open systems, this leads to a quantitative description of the time evolution of classical observables, the most common being the particle density. In the quantum context, diffusive transport has been argued to be intimately connected with Liouvillian dynamics, not just in disordered systems where the correspondence is made with nonlinear σ-models of supersymmetry [6] but also in individual chaotic systems which represent a ballistic limit.

In this paper we present a microwave experiment which demonstrates this deep connection between quantum properties and classical diffusion. Our experiment is a microwave realization of the well-known n-disk geometry, which is a paradigm of an open quantum chaotic system, along with other systems such as the Smale horseshoe and the Baker map [7]. The classical scattering function of the chaotic n-disk system is nondifferentiable and has a self-similar fractal structure. A central property of this fractal repeller is the exponential decay of an initial distribution of classical particles, due to the unstable periodic orbits, which form a Cantor set. This system has received extensive theoretical attention both classically and quantum mechanically. The experimental transmission spectrum directly yields the frequencies and the widths of the low lying quantum resonances of the system [8,9], which are in agreement with semiclassical periodic orbit calculations [10–12]. The same spectra are analyzed to obtain the spectral wave-vector autocorrelation $C(\kappa)$ [8]. The small $\kappa < \gamma_{cl}^{-1}$ (long time) behavior of the spectral autocorrelation is universal in that it is describable within RMT, and provides a measure of the quantum escape rate γ_{qm}, and was previously shown to be in good agreement with the corresponding classical escape rate γ_{cl} [8].

In this paper, we show for the first time that nonuniversal contributions to $C(\kappa)$ for large $\kappa > \gamma_{cl}^{-1}$, which are of the same order as the universal RMT contribution for $\kappa < \gamma_{cl}^{-1}$, can be completely described in terms of the classical RP resonances of the corresponding classical system. Excellent agreement is obtained between the experimental $C(\kappa)$ and the contribution of the classical RP resonances calculated using periodic orbit theory for the n-disk system. Previous work, including our own, had failed to provide a quantitative description of these nonuniversal contributions, which are system specific. In addition to achieving a complete quantitative description of $C(\kappa)$, we are experimentally able to observe the classical RP resonances in a quantum experiment, for the first time.

The experiments are carried out in thin microwave structures consisting of two highly conducting Cu plates about 55×55 cm in area and Cu disks of thickness $d \sim 6$ mm placed between the plates and in contact with them. In order to simulate an infinite system microwave absorber material ECCOSORB AN-77 was sandwiched between the plates at the edges. Microwaves were coupled in and out using terminating coaxial lines which were inserted in the vicinity of the scatterers. All measurements were carried out using an HP8510B vector network analyzer which measured the complex transmission (S_{21}) and reflection (S_{11}) S parameters of the coax + scatterer system. See [9] for details of the experiments. For all frequencies $f < f_c = c/2d = 25$ GHz (here c is the speed of light), Maxwell's equation for the experimental system is identical with the 2D time-independent Schrödinger equation $(\nabla^2 + k^2)\Psi = 0$ with $\Psi = E_z$ the z component of the microwave electric field and $k = 2\pi f/c$. It is this mapping which enables us to study the quantum properties of

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the 2D systems. For all metallic objects between the plates, Dirichlet boundary conditions apply inside the metal. Similar microwave experiments, which exploit this quantum mechanical–electromagnetic (QM-EM) mapping, have been used to study quantum chaos in closed [13,14] and open systems [8].

The transmission function $S_{21}(f)$ which we measure is the response of the system at point $\vec{r}_2$ due to a delta-function excitation at point $\vec{r}_1$, and is determined by the wave function Ψ at the probe locations $\vec{r}_1$ and $\vec{r}_2$. In our experiments the coax lines act as tunneling point contacts, and hence it can be shown [9] that $S_{21}(f) = A(f)G(\vec{r}_1, \vec{r}_2, k)$ is just the two-point Green's function $G(\vec{r}_1, \vec{r}_2, k)$. The scaling function $A(f)$, which represents the impedance characteristics of the coax lines and probes, is sufficiently slowly varying and can be treated as a constant.

The n-disk systems are investigated in the fundamental domain [10], as shown in the inset of Fig. 1, with angles 90° ($n = 2$), 60° ($n = 3$), and 45° ($n = 4$). A typical trace for the three-disk system is shown in Fig. 1. See [9] for details of the comparison of the resonances between experiments and semiclassical calculations. In this paper we focus on the analysis of the spectral autocorrelation of spectra exemplified by Fig. 1.

The spectral autocorrelation function was calculated as [9] $C(\kappa) = \langle |S_{21}[k - (\kappa/2)]|^2 |S_{21}[k + (\kappa/2)]|^2 \rangle_k$. The average is carried out over a band of wave vector centered at certain value k_0 and of width Δk. Since the transmission function is the superposition of many resonances, $|S_{21}(k)|^2 = \sum_i c_i k_i'/[(k - k_i)^2 + k_i'^2]$, with $k_i + ik_i'$ the semiclassical resonances and c_i the coupling which depend on the location of the probes, we have [15]

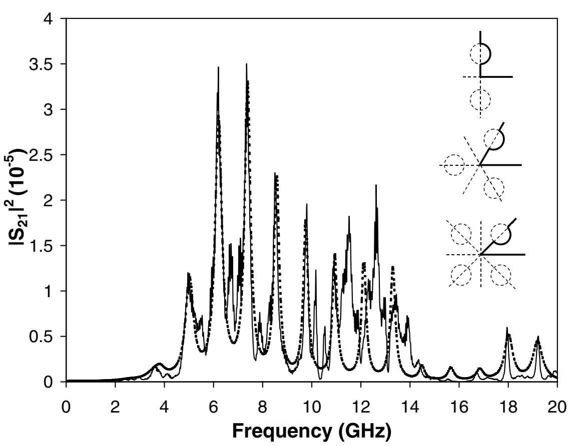

FIG. 1. Transmission spectrum $|S_{21}(k)|^2$ of the three-disk system in the fundamental domain with disk separation $R = 20\sqrt{3}$ cm and radius $a = 5$ cm. The dashed line is the semiclassical calculation. See Ref. [9] for details. Insets: Geometry of the n-disk open billiard ($n = 2, 3, 4$). The solid lines represent the fundamental domain in which the present experiments were carried out.

$$C(\kappa) = \pi \sum_{i,j} \frac{c_i c_j (k_i' + k_j')}{[\kappa - (k_i - k_j)]^2 + (k_i' + k_j')^2} . \quad (1)$$

According to semiclassical theory [16–19], for small κ, the above sum can be replaced by a single Lorentzian

$$C(\kappa) = C(0) \frac{1}{1 + (\kappa/\gamma)^2} , \quad (2)$$

where $\gamma = \gamma_{cl}$, the classical escape rate with the velocity of the classical particles scaled to 1. The above equation was used to fit the spectral autocorrelation for small κ and thus obtain the value of the experimental escape rate γ_{qm} [8,9]. Good agreement of the escape rate is obtained between γ_{qm} obtained from the experiments and γ_{cl} of the classical theory [9].

For intermediate $\kappa > \gamma_{cl}^{-1}$, the semiclassical prediction of Eq. (2) fails because of the presence of the periodic orbits, which leads to nonuniversal behavior. Nonuniversal contributions can play in general a crucial role in determining the overall structure of the spectral autocorrelation, since they can be of the same order of the universal result of RMT. Here we show that the complete $C(\kappa)$ for all κ can be described in terms of classical resonances.

Recently, Agam [20] derived a semiclassical theory to build the connection between the nonuniversality of the spectral autocorrelation and the classical RP resonances. Consider the quantum mechanical propagator [21]

$$K(\vec{r}_1, \vec{r}_2, t) = \frac{1}{2\pi\hbar i} \int G(\vec{r}_1, \vec{r}_2, \sqrt{2m\varepsilon/\hbar^2}) e^{-i\varepsilon t/\hbar} \, d\varepsilon , \quad (3)$$

with $\varepsilon = \hbar^2 k^2/2m$. The integration is performed around $\varepsilon_0 = \hbar^2 k_0^2/2m$, with $\Delta\varepsilon = \hbar v \Delta k$ and $v = \hbar k_0/m$ is the group velocity of the classical particle. The integration in the ε space can be changed into that in the k space as $K(\vec{r}_1, \vec{r}_2, t) = (v/2\pi i) e^{-i\varepsilon_0 t/\hbar} \int_{\Delta k} G(\vec{r}_1, \vec{r}_2, k_0 + k) \times e^{-ivkt} \, dk$. The particle density is $\rho(t) = |K(\vec{r}_1, \vec{r}_2, t)|^2$. The autocorrelation of the particle density is $C_\rho(\tau) = \langle \rho(t)\rho(t + \tau)\rangle_t - \langle \rho \rangle_t^2$ with $\langle \rho \rangle_t \equiv \lim_{t\to\infty}(1/T) \times \int_0^T \rho(t) \, dt$. Using the diagonal approximation, we get $C_\rho(\tau) = (\Delta k v^2/4\pi^2 V^2) \int d\kappa \, C(\kappa) e^{-iv\kappa\tau}$. Here V is the volume of the system with $V \to \infty$ for an open system. If one assumes that the above correlation is classical, one has [22,23] $C_\rho(\tau) = \sum_{i=1}^{\infty} 2b_i e^{-\gamma_i v\tau} \cos \gamma_i' v\tau$, where the b_i are the coupling coefficients, $\gamma_i \pm i\gamma_i'$ the RP resonances of the corresponding classical system in wave vector space. Taking the Fourier transform of the above expression $\int d\tau \, C_\rho(\tau) e^{i\kappa v\tau}$, we get

$$C(\kappa) = \sum_{\pm, i=1}^{\infty} \frac{b_i' \gamma_i}{\gamma_i^2 + (\kappa \pm \gamma_i')^2} , \quad (4)$$

with $b_i' = 2\pi V^2 b_i/\Delta k v^3$.

We now turn to the classical dynamics of the system. The classical evolution is described by the Perron-Frobenius operator whose spectrum, known as the RP resonances, can be calculated as the poles of the classical Ruelle ζ-function. For the hard-disk system, the classical Ruelle ζ-function is [24]

$$\zeta_\beta(s) = \prod_p [1 - \exp(sL_p)/|\Lambda_p|\Lambda_p^{\beta-1}]^{-1}, \qquad (5)$$

here, L_p the length of the periodic orbit p, and Λ_p the eigenvalue of the monodromy matrix associated with the periodic orbits. Only the poles of $\zeta_\beta(s)$ with $\beta = 1$ are calculated since they contribute to the RP resonances with the smallest real part γ_i. The escape rate γ_{cl} is the real pole of $\zeta_1(s)$.

For the integrable two-disk system in the fundamental domain, there is only one prime periodic orbit. We have $\zeta_1^{-1}(s) = 1 - t_0$, where $t_0 = \exp[s(R - 2a)]/\Lambda$, and $\Lambda = (\sigma - 1) + \sqrt{\sigma(\sigma - 2)}$, with the disk separation ratio $\sigma \equiv R/a$. The classical scattering resonances are $\gamma_n + i\gamma_n' = (\ln\Lambda \pm i2n\pi)/(R - 2a)$, with $n = 1, 2, \ldots$. The classical escape rate is $\gamma_{cl} = (\ln\Lambda)/(R - 2a)$. For the chaotic n-disk system, there is no analytical expression of the classical RP resonances. Making use of the cycle expansion [25] and the symmetry factorization of the classical Ruelle ζ-function, the RP resonances can be calculated very accurately [24].

With the available analytical expression of the semiclassical resonances for the two-disk system in the fundamental domain [26], $k_n + ik_n' = [2n\pi + i(1/2)\ln\Lambda]/(R - 2a)$ with $n = 1, 2, \ldots$, one can directly check the validity of Eq. (4). Substituting the semiclassical resonances into Eq. (1), one gets the full two-point correlation function as

$$C(\kappa) \propto \sum_{n=-\infty}^{\infty} \frac{b_n}{\gamma^2 + (\kappa + n\gamma')^2}, \qquad (6)$$

where $\gamma = \gamma_{cl}$, $\gamma' = 2\pi/(R - 2a)$. On the other hand, since the RP resonances of the system are $(\ln\Lambda \pm i2n\pi)/(R - 2a)$, if one puts these resonances into Eq. (4), the above expression (6) follows immediately.

The classical RP resonances can also be obtained experimentally by fitting the spectral autocorrelation with Eq. (4). Since the transmission coefficient $S_{21}(f)$ and also the couplings c_i of the quantum resonances depend on the location of the two probes, so do the couplings b_i' in Eq. (4) of the classical RP resonances. The couplings b_i' are chosen to optimize the fitting. Because of the finite range of the experiment spectrum, only the first eight RP resonances with small real part were obtained. The experimental autocorrelation with the theoretical prediction are shown as Fig. 2. The agreement between the experimental RP resonances and the theoretical ones for the two-disk system is 6% for the position γ_i' and better than 30% for the widths γ_i, it is 7% for γ_i' and 11% for γ_i for the three-disk system, and it is 8% for γ_i' and 17% for γ_i for

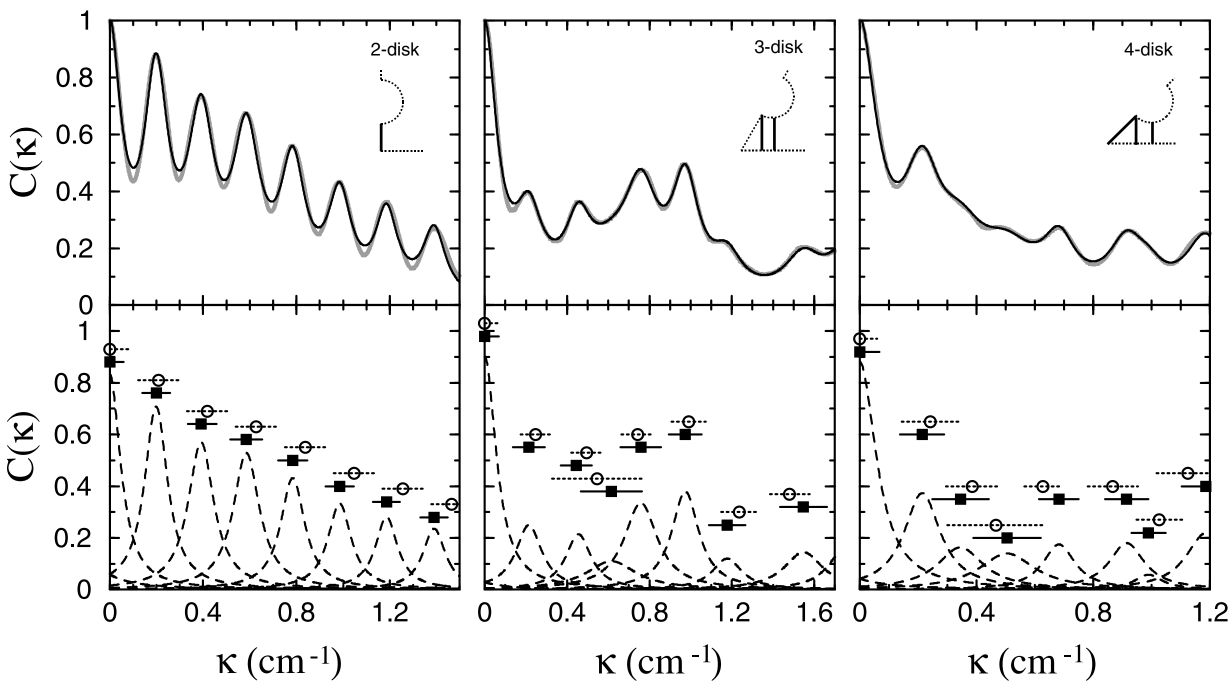

FIG. 2. Autocorrelation function $C(\kappa)$ vs κ (cm^{-1}) of the n-disk system in the fundamental domain. (left) Two-disk system with $R = 40$ cm. (center) Three-disk system with $R/a = 4\sqrt{3}$. (right) Four-disk system with $R/a = 4\sqrt{2}$. In all cases the disk radius $a = 5$ cm. (top panels) Gray line: experimental autocorrelation; solid line: numerical fit to Eq. (4). The insets show the leading periodic orbits (solid lines) for the different geometries. (bottom panels) The dashed lines are the decomposition of the experimental $C(\kappa)$ into Lorentzians. The filled squares are the position γ_i' and the bars the widths γ_i of the experimental RP resonances obtained from the decomposition. The open circles with dotted bars are the positions γ_i' and width γ_i of the predicted Ruelle-Pollicott resonances calculated from Eq. (5).

the four-disk system. We note that these agreements, in particular the wave-vector position γ_i', should be considered as very good. The principal sources for the residual discrepancies are the nonideality of the absorbers, small symmetry-breaking perturbations, and the suppression of some resonances at the neighborhood where the antennas are coupled which affects the autocorrelation function, and therefore the position and the widths of RP resonances. Also very broad resonances are difficult to identify and can lead to an apparent enhancement of the observed widths, which can possibly account for the systematically smaller widths that are observed.

Our investigation clearly demonstrates that the entire quantum spectral autocorrelation for $\kappa > \gamma_{cl}^{-1}$ can be understood completely in terms of the classical RP resonances. The meaning of these RP resonances in the classical context can be understood as follows. If one shoots particles toward the hard disk scatterer, the number of particles that will remain in the scattering region will decay as $N(t) = \sum_i a_i e^{-\alpha_i t}$. Besides the general exponential decay at $\alpha_0 = v\gamma_{cl}$, there are oscillations due to the fact that the RP resonances $\alpha_i = v(\gamma_i \pm i\gamma_i')$ are not always real [24] as contrasted with the purely diffusive system. Taking the Fourier transform of $N(t)$, one can identify the Lorentzians in the spectrum with the RP resonances. Our work demonstrates that suitable quantum correlations diffuse just like classical observables in an open system.

It is remarkable that the same experiment yields both the quantum resonances and the classical RP resonances. Thus we have demonstrated experimentally the profound connection between quantum properties and classical diffusion. This connection is best seen in open quantum systems. We have demonstrated this connection for the model n-disk geometry by combining our experiments with a vast body of theoretical results for the quantum and classical properties that is available for this system. However, we believe the results of this work should apply to arbitrary open chaotic systems. The present results also have wider implications in a variety of phenomena in different fields in physics, such as photodissociation of atoms [27], nuclear decay [28], electronic transport, fluid dynamics, and acoustic and electromagnetic propagation.

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