EXPERIMENTAL EIGENVALUE SPECTRA OF "ROUGH" AND MULTIPLY-CONNECTED BILLIARDS

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We use microwave cavities to study the spectra of billiards with "rough" perimeters, or which are multiply connected. While simple boundaries show cumulative level densities N(E) which are in good agreement with the Weyl formula, qualitative departures are observed at low energies when the perimeter is "roughened" by increasing the ratio of perimeter L to area A, and when obstacles are introduced to make the billiard multiply connected. Thus the Weyl formula even including perimeter corrections is not a good yardstick at low energies which probe internal length scales. Interestingly, in the case of a multiply-connected billiard with a periodic array of obstacles, large "gaps" are observed in the discrete spectrum, somewhat like a solid -state lattice of finite size. The gaps persist when disorder is introduced, i.e. in the analog of a solid-state glass.

The influence of the shape of a planar domain on the eigenvalue spectrum of the Schrodinger equation is well known[1]. In general the eigenvalue spectrum can be represented as $N(E) = N_{sm}(E) + N_{fluc}(E)$, where $N_{sm}(E)$ is a smooth function determined by geometry and topology alone. The nature of the dynamics is contained in the statistical properties of the fluctuating part $N_{fluc}(E)$, and several general results distinguishing between geometries whose classical dynamics is integrable or chaotic have been obtained[2].

The functional form of $N_{sm}(E)$ has long fascinated physicists and mathematicians. The "volume" dependence was obtained rigorously by Weyl[3], who showed that $N_{sm}(E) = (A/4\pi)\,E$ (in 2-D) in the limit of large E. Thus the density of states is independent of the shape in the semiclassical limit. The corrections due to the influence of the shape of the boundary, which is *smooth*, were initially also obtained for large E as

$$N_{sm}(E) = (A/4\pi)\,E - (L/4\pi)\,E^{1/2} + K \qquad (1)$$

where L is the perimeter and K is a constant determined by the curvature and topology. This relation has been found to work extremely well in "smooth" geometries even at low energies, to the extent that it is used as a yardstick for estimating missing levels.

The issue of what happens to the spectrum when the perimeter is *rough*, i.e. has many length scales, has recently received attention. For *fractal boundaries*, an important conjecture was put forth by Berry[4], who suggested that $N_{sm}(E) = (A/4\pi)\, E - c_d\, \mathcal{H}\, E^{d/2} + O(E^{d/2})$ where $\mathcal{H}$ is the Hausdorff measure and d the Hausdorff dimension. This conjecture has stimulated many investigations by mathematicians, and more rigorous arguments have led to improved results[5].

The problem is of interest from several viewpoints, for example, the "inverse" problem in electromagnetics, electromagnetic waves in rough enclosures[6], diffusion in porous media[7] and atomic clusters. In practise, the systems mentioned are usually probed at low energies in order to gain information regarding internal structure and length scales. Thus it is useful to enquire as to the applicability at low E of asymptotic formula such as the ones mentioned above.

Numerical calculations are usually needed to actually obtain quantitative results for low energies and for real geometries which are often of interest. However solutions of the wave equation can only be carried out for a few simple geometries - even chaotic geometries such as the stadium and the Sinai billiard require special computational techniques to obtain several hundred eigenvalues. Approaches using actual physical experiments have only recently begun addressing these issues. In acoustic experiments[8] absorption appears to play an important role leading to some puzzling effects. However microwave experiments[9,10] have recently been shown to yield very high quality results for both eigenvalues and eigenfunctions, having led among other results, to the observation[9] of scars in wavefunctions, and the verification[11] of the isospectral theorem.

This paper utilizes 2-D microwave cavities to explore the eigenvalue spectrum in geometries where multiple length scales are present, and the geometries may be singly or multiply connected. The basis of the experiments is now well established, viz. that for very thin cavities of thickness d, for frequencies $f < c/2d$, the electromagnetic waves in the enclosure obey the 2-D Helmholtz-Schrodinger equation : $(\nabla^2 + k^2)\Psi = 0$, with Dirichlet boundary conditions. In our experiments the eigenfrequencies f_n are measured as transmission resonances, and energy eigenvalues are obtained from $E_n = k_n^2 = (2\pi f_n / c)^2$. Further details of the experimental techniques are described in ref.12. A particular advantage

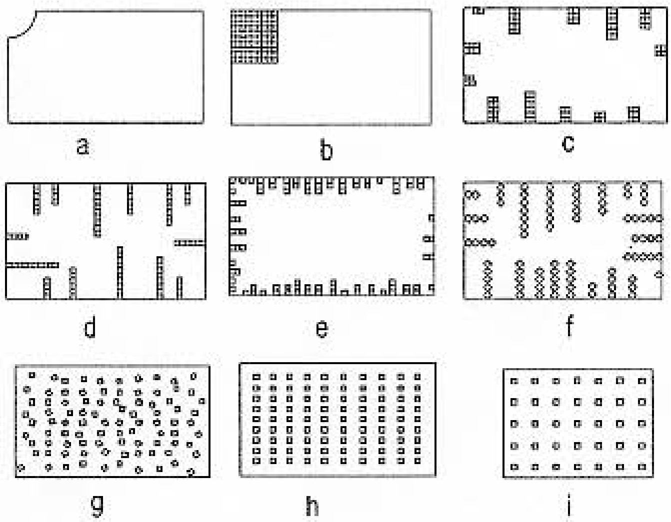

FIGURE 1. Crossections of the cavities studied. The areas A(cm^2) are : (a) 939.6, (b) - (h) 869.2, (i) 636.4. The perimeters L(cm) are : (a) 129.5, (b) 131.6, (c) 221.6, (d) 309.6, (e) 309.6, (f), (g), (h) 399.6, (i) 245.2

of the experiments is the ability to study arbitrary geometries, as will be apparent from the subsequent discussion.

A sequence of cavities whose crosssections are shown in Fig.1, was studied in which the ratio L/A and the connectivity were systematically varied. All material was made of Cu and in all cavities d = 6 mm. In the frequency range upto 20 GHz, typically 900 levels were observed. Besides the more standard geometries of a rectangle and a Sinai billiard (Fig.1(a)), which may be regarded as smooth, a set of cavities (Fig. 1(b) - (f)) was fashioned from a parent rectangle of dimensions 44 x 22 cm., and a set of 90 square tiles of side 1 cm. The tiles could be arranged to vary L/A by starting with them all in one corner to form a psuedo-integrable billiard (Fig.1.b) and progressively increasing L (keeping A fixed) by arranging the tiles in the form of "fingers" along the rectangular perimeter (Fig. 1 (d) - (f)). Finally the tiles can also be arranged on periodic "lattices" (Fig.1(h, i) or placed randomly to fashion a disordered "glass" (Fig.1(g)).

SPECTRA OF "SMOOTH" BILLIARDS

The experimental staircase density of states $N(\hat{E})$ vs. $\hat{E}$

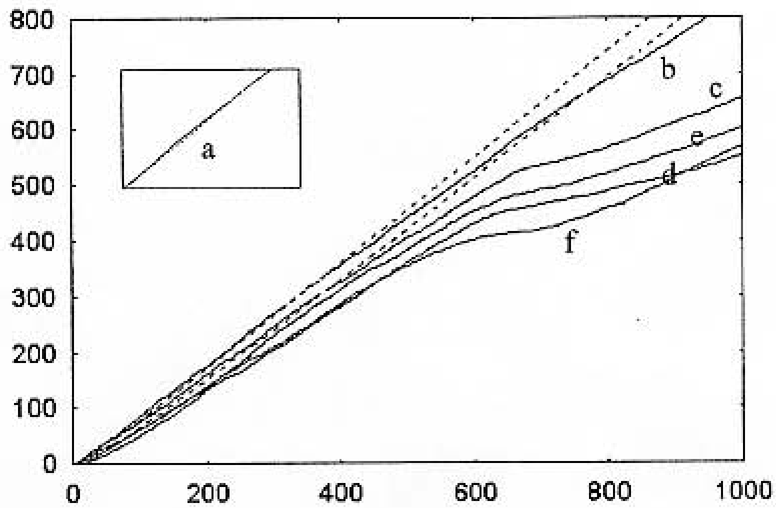

FIGURE 2. N(Ê) vs. Ê (= AE/4π) for the cavities shown in Fig.1(a) - (f), along with comparisons to eqn.(1) including perimeter and curvature corrections. The inset shows the data for the cavity (a). The dashed lines in the main figure represent the range of eqn.(1) for the cavities (b) - (f).

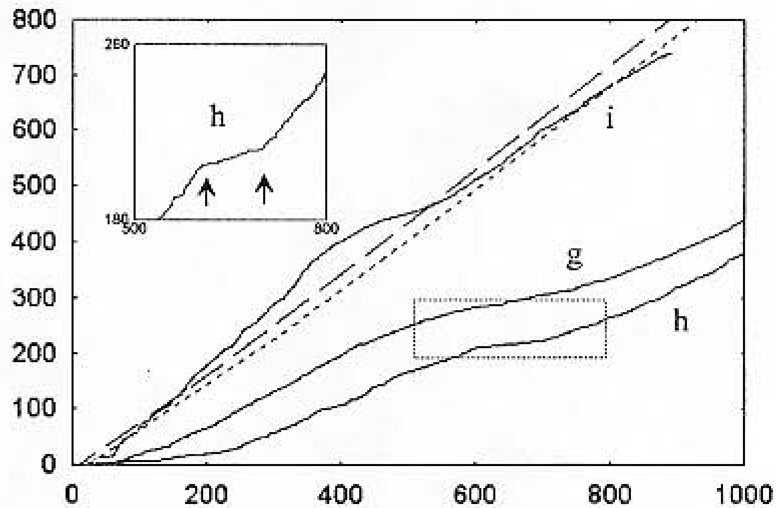

FIGURE 3. N(Ê) vs. Ê for the multiply-connected cavities (g) to (i) shown in Fig.1. The dashed line represents eqn.(1) for cavity (i) and the dotted line for the other two. The inset is a magnification of the dashed box region in the main figure for cavity (h), and shows the region of sparse states, bracketed by the arrows. The arrows correspond in frequency to 14.1 and 15.2 GHz.

(=EA/4π) of a quarter Sinai billiard is compared with the corresponding Weyl formula in the inset to Fig.2. It is evident that the agreement is excellent over the wide range of energies accessible in the present experiment. We have earlier shown[13] that experiment and corresponding numerical simulations agree very well as regards individual eigenfunctions, and eigenfrequencies. Indeed it is important to emphasize that if adequate care is taken regarding the measurements, then for the chaotic geometry very high quality results are obtained. In forthcoming work, it will be shown that detailed statistical analysis of the spectrum as regards various measures (eg. spacing statistics, etc) are also in good agreement with theoretical expectations.

SPECTRA OF "ROUGH" BILLIARDS

On the other hand, the experimental $N(\hat{E})$ for the cavities shown in Fig.1(b) - (f) which is diplayed in Fig.2 show clear departures from the Weyl formula at the low to intermediate energies of the present measurements. As L/A is increased, the substantial decrease below the Weyl formula is evident. It is important to emphasize that the curvature correction K, which is between 0.208 and 4.454 for cavities (a) to (f) in Fig.1, and $K = -12.25$ for the cavities (g) and (h), and which is important in comparisons of experiment and theory for smooth billiards, is inadequate for explaining the deviations between experiment and theory in the present case. The feature which occurs around $\hat{E} \sim 600 - 700$ in the curves in Fig.2, cannot be immediately attributed to a dominant length scale in the different geometries, such as the tile size or distances between "fingers". Also the pronounced deviations from the corresponding Weyl expressions (dashed lines in Fig.2) are suggestive of oscillatory contributions, and cannot be described by fractal descriptions of the boundaries as represented by say the Weyl-Berry conjecture. A more quantitative theory of such effects is clearly needed.

SOLID-STATE LATTICE AND GLASS : "GAPS" IN DISCRETE SPECTRA

Particularly interesting results are obtained when the cavities are made multiply connected by placing the tiles inside. When a square lattice of period 4.4 cm. is formed (Fig.1 (i)) with appropriate spacing of 2.2 cm from the walls to enable reflection to mimic an infinite system, the resultant density of states only approximately tracks the Weyl formula (Fig.3). However in this case the departures from the Weyl formula can be understandable in terms of ideas from solid-state physics[14,15]. Internal reflection between tiles, much like Bragg reflection between atoms, leads to interference. Just as the energy spectrum of atoms on a periodic lattice deviates from the free particle relation via the manifestation of gaps, large "gaps" are observed in the spectrum of the multiply connected billiards. This is most pronounced in the geometry of Fig.1(h), but is also present in the geometry of Fig.1(i), as can be seen from the $N(\hat{E})$ displayed in Fig.3. The data for cavity (i) represent level accumulation at energies below the "gap" location, and level depletion at slightly higher energies. The location of this feature at $\hat{E} \sim 600$ is well correlated with the minimum frequency for waves (~ 12 GHz) to form standing wave patterns between the square tiles.

Interestingly a manifestation of the gap continues even when the periodicity is destroyed by placing the squares as randomly as possible. This system (Fig. 1(g)) is an analog of a solid-state "glass ", and also may bear similarities to atomic clusters. Essentially structure is apparent near the same location in energy even in the disordered system (g) as in the corresponding periodic system (h), with anomalously increased level spacing present locally in comparison to the mean spacing at other energies.

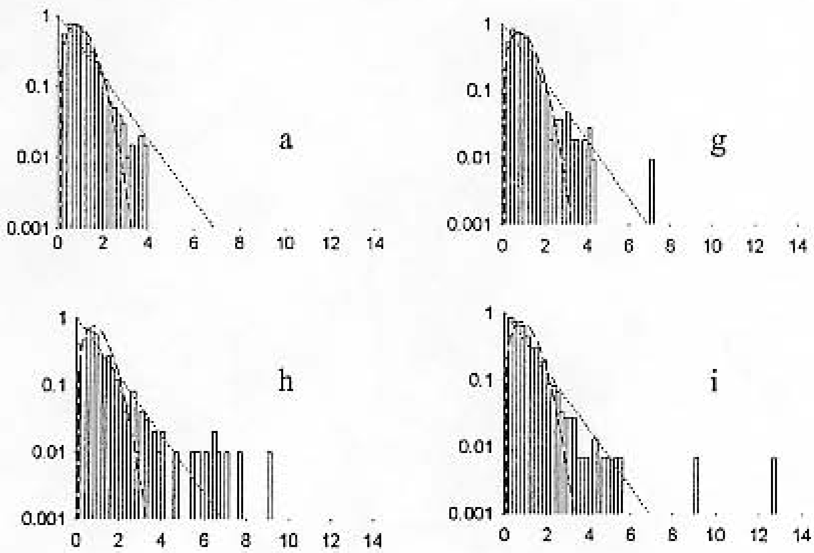

FIGURE 4. Spacing statistics P(s) vs. s for some of the cavities. The vertical logarithmic scale is used to emphasize the presence of levels at large spacing s > 4, and which are absent in the Sinai billiard (a). The lines are Poisson (dotted) and Wigner (dashed) distributions.

The presence of large "gaps" can be demonstrated by examining the spacing statistics at large spacings. This is shown in Fig.4 for the "smooth" cavity (b) and for the multiply connected cavities (g) to (i). The logarithmic scale for P(s) is used to emphasize the presence of levels with nearest-neighbor spacings from 4 to 14 times the mean spacing (which is nearly the same for all the cavities), well above the predictions of the Poisson and Wigner distributions also shown, in both the periodic and the glassy cavities. The large spacings for $s = 10$ to 14 for cavity (h) represent the visible "gaps" in N(E) in the inset to Fig. 3.

A puzzling feature which was observed experimentally was the rather low Q factors ($< 10^3$) for some of the multiply connected cavities, particularly the square lattice. Since the area exposed to the metal is not significantly changed, this is unlikely to be due to absorption in the metal, which is the dominant cause in the smooth cavities where $Q \sim 10^4$. More likely this is a manifestation of broadening due to internal reflection.

The analogy between electromagnetic waves and electrons in solids, in particular the presence of a "photonic" gap and disorder introduced states in the gap, has been studied in other experiments[16] also. However the experiments described in this paper are closer to the ideal problem of an electron in a periodic potential, than those experiments which use dielectric scatterers arranged on a lattice. Also our experiments are in finite systems rather than open transmission experiments. An aspect which needs to be studied is the possibility of localization of wavefunctions, which are accessible in our experiments and is the focus of future work. An important question is to examine whether the results obtained for Anderson localization in disordered systems is applicable to the "glass" geometry (g).

In conclusion, the work reported here demonstrates that physical experiments using electromagnetic systems can study geometries which may be inaccessible by computational means. The results for the eigenvalue spectra show strong deviations from the asymptotic relations such as the Weyl formula due to internal scattering occuring at the internal length scales. While qualitative understanding is possible in some cases, such as the periodic lattice and which may also be amenable to theoretical treatment, a general quantitative explanation of the results does not appear to be available, and remains a theoretical challenge. It is however apparent that the microwave experiments afford excellent, clean, model systems for studying the quantum mechanics of non-interacting electrons in a variety of situations, from solid state physics to atomic clusters.

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