

## Experimental Observation of Scarred Eigenfunctions of Chaotic Microwave Cavities

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The wave functions of Sinai-billiard-shaped microwave cavities are experimentally studied. Some of the general features observed are parity breaking in the lowest eigenstates, "bouncing-ball" states, and states with quasirectangular or quasicircular symmetry. The above features are associated with nonisolated periodic orbits. Some states are observed which can be associated with isolated periodic orbits, leading to scars. This work represents the first direct experimental observation of scarred eigenfunctions. At high frequencies the eigenfunctions are very complex, and are yet to be classified in a general scheme.

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The quantum-mechanical (QM) behavior of systems whose classical dynamics is chaotic has been the subject of considerable recent interest [1,2]. A model system often studied theoretically is the two-dimensional billiard, such as the Sinai billiard and the Bunimovich stadium. Of interest are the eigenvalues and the eigenfunctions of such geometries. The nonintegrable nature of the problem precludes analytical results and one must necessarily resort to numerical computation. Some general features of the eigenvalue spectrum, such as its statistics, are understood [3], and have even been compared with experiments. Regarding the eigenfunctions, while significant progress has been made in numerical computation [4] using high-speed computers, the general features are much less understood, except for the important association with periodic orbits, leading to the so-called "scars" [5]. To date the eigenfunctions have been inaccessible to experiment, and although indirect evidence of the consequence of scarred eigenfunctions on atomic spectra have been reported [6], there has been no direct experimental validation of theoretical concepts, or even of the numerical calculations.

In this paper, I report the direct experimental observation of the eigenfunctions of microwave cavities shaped in the form of 2D Sinai billiards, consisting of a circular disk inside a rectangle. The technique, which employs a cavity perturbation method, enables the mapping of the wave function over the entire region, in both singly and multiply connected cavities. A direct "snapshot" of the eigenfunction is obtained. To my knowledge, this is the first experimental observation of the eigenfunctions of cavities which enables the complete spatial  $(x,y)$  dependence to be clearly observed.

The close analogy of certain microwave (and acoustic) systems to quantum problems has long been recognized. For both a QM particle and the electromagnetic (EM) field in a cavity, the same equation applies, viz.  $(\nabla^2 + k^2) \times \Psi(x,y) = 0$ , with  $\{k^2 \rightarrow \omega^2/c^2, \Psi \rightarrow \text{electric field } E_z\}$  for the EM case, and  $\{k^2 \rightarrow 2mE/\hbar^2, \Psi \rightarrow \text{wave function } \psi\}$  for the QM case. In both cases,  $\psi$  or  $E_z$  vanishes on the boundary. For a cylindrical cavity (of arbitrary cross section in the  $x-y$  plane), there exists a set of transverse magnetic modes with uniform spatial variation of

the fields on the  $z$  direction, which obey the above two-dimensional equation. The correspondence of the two-dimensional case has been used to study eigenvalue spectral statistics in billiard-shaped cavities [7], and also for chaotic microwave scattering geometries [8,9].

The results reported here are primarily for Sinai-billiard-shaped cavities, consisting of a rectangle of dimensions  $21.8 \text{ cm} \times 44 \text{ cm} \times 6 \text{ mm}$ , with a solid circular disk  $10.15 \text{ cm diam} \times 6 \text{ mm}$ , placed either at the center of the rectangle (this configuration is called Si-2) or at one corner (called Si-1). We also compare the results with data for the parent rectangle ( $21.8 \text{ cm} \times 44 \text{ cm} \times 6 \text{ mm}$ ). The small  $z$  dimension of  $6 \text{ mm}$  ensures that at frequencies  $f < c/(1.2 \text{ cm}) = 25 \text{ GHz}$ , all the modes observed are of the type mentioned above. Cavities were operated [10] in a transmission mode, with transmission spectra obtained using an HP8510 Network Analyzer, usable up to  $20 \text{ GHz}$ . Input and output coupling to the cavity *magnetic fields* was achieved with loops connected to semirigid coaxial cable. Cavity resonances are easily identified as resonance peaks in the transmission spectrum. We have carried out a separate detailed study of the eigenvalue statistics which will be reported elsewhere. In this paper we focus only on the *eigenfunctions*.

The cavity fields were mapped out using a cavity perturbation technique [11]. Here a small *metallic* perturbing object (bead) causes a shift  $\Delta f$  of the resonant frequency  $f_0$  proportional to the local field density  $E_z^2$  given by  $\Delta f/f_0 = -\int E_z^2 dV_b / \int E_z^2 dV_c$ , where the subscript  $b$  ( $c$ ) refers to integrations over the bead (cavity) volumes, respectively. For objects much smaller than the wavelengths involved, the expression can be simplified to

$$\Delta f(x,y) = -kE_z^2(x,y). \quad (1)$$

Here  $k$  is a geometric factor which is of course mode and bead specific. For a given mode, Eq. (1) enables the determination of the normalized local-field intensity by measurement of the frequency shift for different locations of the bead. Conceptually the technique is similar to mass loading of a drum head and listening to the timbre. The bead is a spherical metallic sphere, of typical diameter  $1$  or  $2 \text{ mm}$ . All earlier implementations [12] to date of this method have been one dimensional. Here, for the

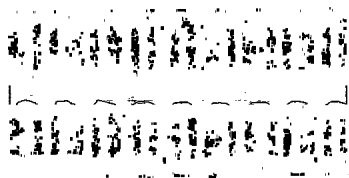


FIG. 1. Field profile for the  $f=3.360$  GHz mode of the rectangular cavity. The contour plots were obtained from a measurement grid of  $42 \times 20$  points.

first time, we have implemented the technique in two dimensions also. In the experiment, it is possible to obtain data for between 840 points ( $42 \times 20$  grid) and 13440 points ( $168 \times 80$  grid) in 1 to 4 h.

As proof of the validity of the experimental technique, we display in Fig. 1 the field profile for the 3.360-GHz mode [13] of the bare rectangular cavity. The data are presented in the form of  $x$ - $y$  contour plots. Without ambiguity, it is obvious that the data obey the expected  $E_z^2 \propto \sin^2(p\pi x/a)\sin^2(q\pi y/b)$  form obtained analytically, with  $p=2$ ,  $q=9$ . The calculated (3.363 GHz) and measured frequency agree to  $<0.1\%$ . Several other modes of the rectangle studied were all consistent with theoretical expectations, without exception.

For the Sinai billiards Si-1 and Si-2, about a hundred modes were examined. The data show a rich behavior of the field profiles, almost overwhelming in the diversity observed. We describe below certain general features which we have extracted.

**Parity breaking.**—A striking feature of some of the eigenfunctions is breaking of the left-right symmetry, particularly in the lowest modes, due to slight asymmetries. The ground state of Si-2 is shown in Fig. 2(a). It consists of two “bubbles” on either side of the disk, which are slightly asymmetric in field strength due to the slight ( $<0.1\%$ ) inaccuracy in centering the disk. Upon moving the disk to enhance the horizontal asymmetry to 1%, the ground state splits into two modes, which are predominantly localized in the left [Fig. 2(b)] or right (not shown) regions. Interestingly, this situation is exactly analogous to the 1D quantum problem of an infinitely deep square-well potential with a barrier of finite height and width at the center. The central perturbation in both cases is highly singular as regards its parity-breaking effects.

**“Bouncing-ball” states.**—Some of the experimentally observed modes have a character [Fig. 3(a)] which is clearly identifiable with the “bouncing-ball” states first discovered by McDonald and Kaufman [14] in numerical simulations of a stadium. This mode appears to *avoid* the circular central disk, and is confined to the region between the disk and the rectangle. A natural question that arises is whether the frequency of this mode can be calculated. Indeed, the measured frequency of 3.529 GHz is intermediate between the calculated frequency (3.554

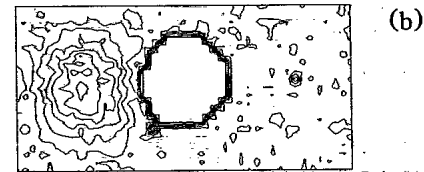
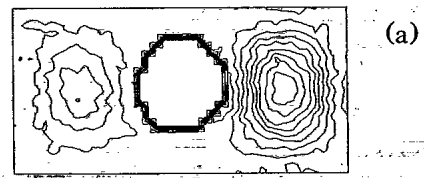


FIG. 2. The ground-state eigenfunction of Si-2, illustrating the effects of parity breaking. (a)  $f=1.081$  GHz,  $n=1$ , with asymmetry  $<0.1\%$ . (b) Splitting of the ground state into left ( $f=1.076$  GHz) and right ( $f=1.096$  GHz, not shown) modes due to an asymmetry of about 1%. (In this and other figures, the apparent distortions around the disk are artifacts of the contour plotting procedure and the finite measurement grid, and should be ignored.)

GHz) for the  $5 \times 2$  mode of a rectangle  $21.8 \text{ cm} \times 34 \text{ cm}$  (i.e., ignoring the central region containing the disk) and the calculated value for the empty rectangle (3.507 GHz).

**Eigenstates with quasirectangular symmetries.**—Another class of modes has a clearly recognizable quasirectangular behavior, whose frequencies are in close agreement with those of the  $2 \times q$  modes of the unperturbed rectangle. An example is shown in Fig. 3(b). The frequency of the  $2 \times 11$  mode of the bare  $21.8 \text{ cm} \times 44 \text{ cm}$  rectangle is calculated to be 3.995 GHz, close to the observed frequency of 4.074 GHz. Several other modes also have the same  $2 \times q$  character. In all these cases, the calculated and observed frequencies are in fairly close agree-

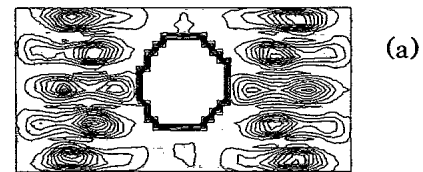


FIG. 3. (a) Bouncing-ball state of Si-2,  $f=3.529$  GHz,  $n=28$ . (b) Quasirectangular mode of Si-2,  $f=4.074$  GHz. This mode corresponds to the  $2 \times 11$  mode of the empty rectangle.

ment, which implies that these modes essentially *ignore* the presence of the disk. We make a distinction between these and the bouncing-ball states mentioned above, since, in our view, the latter *avoid* the disk. (These quasirectangular states are also observed in Si-1.)

*States associated with isolated periodic orbits.*—It is generally believed that periodic orbits (PO) play an important role in determining the eigenvalues and the wave functions. In particular, Heller [5] has shown that PO lead to scars, which manifest as excess density along the PO.

In the Sinai billiard Si-2, several simple PO can be identified. In certain modes, the intensity contours can be associated with some of the PO. Examples are shown in Fig. 4, in which the associated PO are also inscribed.

Some interesting features of these scarred wave functions are worth noting. In Fig. 4(a), the associated PO are the half diagonal lines, while in Fig. 4(b) the PO are diagonal. We have not yet identified states associated with other PO, e.g., the one joining the left and right sides of the rectangle with the circle. One expects that the relative importance of various PO is determined by the associated [5] Lyapunov exponents, and future work will explore this connection qualitatively and in greater detail.

While the qualitative association with a PO is easily carried out, the calculation of the frequency *using simple rules* is not so straightforward, and may be impossible. Use of a 1D quantization utilizing the path length for the PO does not lead to very close correspondence for the measured and calculated frequencies. For instance, in Fig. 4(b), taking the path length as 19.6 cm, and taking this to be associated with four half wavelengths, we arrive at a frequency of 3.06 GHz, in comparison with the measured value of 3.663 GHz. This difference arises because the frequency is essentially determined by 2D quantization, while the ray approach is 1D.

The eigenstates in Fig. 4 are all associated with isolat-

ed PO, in contrast to the states discussed earlier (Figs. 2 and 3), in which the PO are not isolated. The present experiment is the first direct experimental realization of the association of the wave functions with classical PO, and of scars.

*Modes of the Sinai billiard Si-1.*—Since our experimental method can easily handle any other geometry, we have also studied another configuration (called Si-1), in which the disk is placed at the top left-hand corner. This geometry does not have any symmetries, and would be difficult to study using purely numerical methods. While we have examined several modes in this configuration, the lack of identifiable symmetries produces very complex behavior, much of which remains to be understood. Up to now we have identified only some of the modes, one class of which is the quasirectangular modes mentioned earlier in the discussion of Si-2.

Another class of modes has a *diagonal character*, and can be associated with the PO along the diagonal. One member of this family is shown in Fig. 5(a), and has ten maxima predominantly along the diagonal. Attempting a 1D quantization along the diagonal length 48.6 cm, we arrive at a calculated value of 3.086 GHz, compared to the experimental value of 3.138 GHz. Other members of this family occur at 2.487 GHz (seven maxima) and 2.810 GHz (eight maxima) with calculated values of 2.16 and 2.469 GHz. The correspondence appears to improve with frequency. It should be noted that this PO is not isolated, in that slight displacements leading to PO in the form of parallelepipeds occur around it. Thus the diagonal PO indicated in Fig. 5(a) should be regarded as indicative, and not exact.

We have hitherto discussed clearly identifiable features of several low-lying states of the Sinai billiards Si-1 and Si-2. At higher frequencies, the eigenstates are quite complex [an example is shown in Fig. 5(b)], and associations with simple isolated or nonisolated PO is not obvious. Future work will concentrate on this aspect.

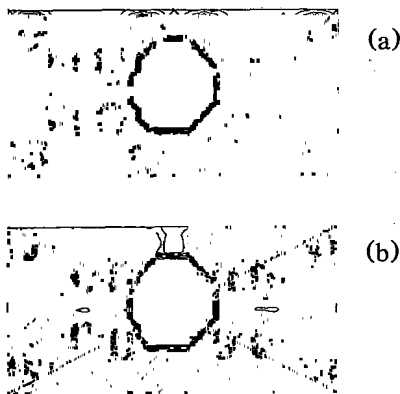


FIG. 4. Scarred eigenstates of Si-2. The associated PO is inscribed as straight lines. (a)  $f=3.112$  GHz,  $n=20$ ; (b)  $f=3.663$  GHz,  $n=30$ .

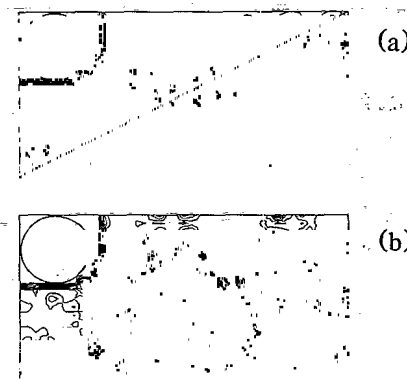


FIG. 5. (a) Scarred eigenstates of Si-1,  $f=3.138$  GHz. The principal PO, which are along the diagonal, are inscribed. (The disk is in the top left corner.) (b) Complex eigenstates of Si-1,  $f=8.348$  GHz.

Whether every state can be associated with one or several PO remains to be studied. Ultimately the challenge would be to obtain the frequency from general rules, such as are possible in the regular (e.g., rectangle, circle) geometries, and appear to hold approximately for some eigenstates in the billiards.

In the regular rectangle geometry,  $k_x$  and  $k_y$  are good quantum numbers, while for the irregular billiard, a doubly denumerable set of quantum numbers is not easily identifiable, and possibly does not exist. However, the discussion above suggests that for at least for some families of eigenfunctions, such as those in Figs. 2 and 3 associated with nonisolated PO, it may be possible to identify a set of "nearly good" quantum numbers. For the states associated with isolated PO, such an identification appears to be questionable.

In addition to its importance as an experimental realization of a model quantum chaotic system, the microwave experiments reported here also provide a new technique for determining field profiles in cavity geometries used in plasma physics and accelerator structures. The results show that unusual field profiles are obtained even in simple geometries, and raise the possibility of tailoring field geometries to suit experimental needs in these areas. The Sinai-billiard geometry studied here also forms the unit cell of a lattice model often considered in electron quantum transport. The results obtained here on a macroscopic scale can be easily scaled down to equivalent systems in atomic and mesoscopic physics.

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